Beilinson's conjecture for the Dwork family.

(joint work in progress with Lambert A'Campo)

0. Motivation. (n even)

Dwork family: $Y_t : X_0^{n+1} + X_1^{n+1} + \dots + X_n^{n+1} = (n+1) \cdot t \cdot X_0 \dots X_n$ in $\mathbb{P}^n$

[**E.g.** $t = 0 \rightsquigarrow$ Fermat hypersurface
$n = 2 \rightsquigarrow$ univ. family of elliptic curves with 3-torsion]

Group actions: $G = \mu_{n+1}^{n+1} / \Delta\mu_{n+1} \supseteq H = \{ \zeta \in G : \prod \zeta_i = 1 \}$

$G \circlearrowright (Y_0)_{\mathbb{Q}(\zeta_{n+1})}$, $\quad H \circlearrowright (Y_t)_{\mathbb{Q}(\zeta_{n+1})}$ any t

$\rightsquigarrow$ consider $M_t := H^{n-1}(Y_t)^H$ motive of H-invariants / $\mathbb{Q}$.

Thm (Harris – Shepherd-Barron – Taylor).
M_t is "potentially automorphic", i.e. $\exists$ extension $K/\mathbb{Q}$ and an automorphic rep. π of $GL_{n,K}$ s.t. $L(M_t, s) = L(\pi, s)$.
In particular, $L(M_t, s)$ has analytic cont. & functional eqn.
$$L(M_t, s) \sim L(M_t, n-s).$$

E.g. $n=2$: mod. forms wt 2 & $L(f, s)$
$n=4$: can strengthen this to π on $GSp_{4,K}$ & $L(\pi, \mathrm{spin}, s)$

Goal. Study Beilinson's conjecture for $L(M_t, s) = L(\pi, s)$:

$$L(M_t, n) \sim L^{(n)}(M_t, 0) \overset{??}{\longleftarrow\dashrightarrow} H^n_{\mathcal{M}}(M_t, \mathbb{Q}(n)) = CH^n(M_t, n) = K_n(M_t)^{(n)}_{\mathbb{Q}}$$
(conjecturally dim n).

Rmk. Related work of Doran-Kerr constructs $\underline{\underline{\text{one}}}$ class.

Strategy.

1. $t=0$ $\xrightarrow{\text{deform!}}$ 2. $t\neq 0$

$Y_0: x_0^{n+1}+\dots+x_n^{n+1}=0 \circlearrowright G$

$\rightsquigarrow$ construct enough classes

$\alpha_0 \in H^n_{\mathcal{M}}(Y_0, \mathbb{Q}(n))$

HC $Y_t: x_0^{n+1}+\dots+x_n^{n+1}=(n+1)t x_0\dots x_n$

$\rightsquigarrow$ find deformation α_t of α_0

$t\frac{d}{dt}\,\mathrm{reg}(\alpha_t)\sim$ periods of Y_t

$\Rightarrow$ integrate & use $t=0$ to get constant term

1. Milnor K-theory of Fermat hypersurfaces.

In general, consider the degree d Fermat hypersurface:

$$F_n^d:\ x_1^d+\dots+x_n^d=x_0^d \quad \text{in } \mathbb{P}^n.$$

$$\circlearrowleft$$

$$G_d:=\mu_d^{n+1}/\Delta\mu_d \qquad \hat{G}_d\cong(\mathbb{Z}/d\mathbb{Z})^n\ni \underline{a}$$

representatives $a_i\in\{1,\dots,d\}$

$$\Rightarrow\ H^{n-1}(F_n^d,\mathbb{C})_{\mathrm{prim}}=\bigoplus_{\substack{\underline{a}\text{ s.t.}\\ a_i\neq 0\\ \sum a_i\neq 0}}\underbrace{H^{n-1}(F_n^d,\mathbb{C})_{\underline{a}}}_{\text{basis: }\omega_d^{\underline{a}}}$$

We consider the class:

$$\alpha_0:=\left\{1-\frac{x_1}{x_0},\dots,1-\frac{x_n}{x_0}\right\}\in K_n^M(\mathbb{Q}(F_n^d))\otimes\mathbb{Q}.$$

& Kerr's regulator on Milnor K-theory reg^M.

Def. For $\underline{\alpha}=(\alpha_1,\dots,\alpha_n)\in[0,1]^n$, define

$$\tilde{F}(\underline{\alpha}):=\sum_{\underline{i}\in\mathbb{Z}^n_{\geq 0}}\frac{\Gamma(\alpha_1+i_1)\cdots\Gamma(\alpha_n+i_n)}{\Gamma(\alpha_1+\dots+\alpha_n+i_1+\dots+i_n+1)}.$$

Rmk. • $\tilde{F}(\underline{\alpha})$ = value at $(1,\dots,1)$ of Lauricella's multivariable hypergeometric series, called F_B.

• Previously, thought it can be expressed in terms of the single variable hypergeometric series ${}_{n+1}F_n$, but it doesn't seem so.

Thm (A'Campo–H.). $\alpha_0 \in K_n^M(\mathbb{Q}(F_n^d))$ as above. Then on the analytic open set $\mathcal{U} := \{X_0 = 1, |X_i| < 1\}$, we have:

$$\mathrm{reg}(\alpha_0) = \frac{(-1)^n}{d^2} \sum_{\underline{a} \in (\mathbb{Z}/d\mathbb{Z})^n} \tilde{F}\left(\frac{a_1}{d}, \dots, \frac{a_n}{d}\right) \cdot \omega_d^{\underline{a}}.$$

(Note: basis of $H_{n-1}(F_n^d(\mathbb{C}), \mathbb{Z})$ restricts non-trivially to $\mathcal{U} \Rightarrow$ this is enough.)

Rmks: For $n = 2$: [K_2(curve), rank 1]

• due to Otsubo (Crelle, 2011)

• deformation to $t \neq 0$ due to Nemoto (preprint, 2023).

2. Consequences for Beilinson's conjecture. [$K_{>2}(\dim > 1)$, rank ≥ 1]

Technical issue:

Have: | Beilinson's conjecture.

• $\alpha_0 \in K_n^M(\mathbb{Q}(F_n^d)) \rightsquigarrow [F_{\alpha_0}] \in CH^n(F_n^d \setminus \mathrm{div}(\alpha_0), n) \overset{??}{\rightsquigarrow}$ • $\beta \in CH^n(F_n^d, n)$

• $\mathrm{reg}^M(\alpha_0)\big|_{\mathcal{U}} = \dots \overset{??}{=} \mathrm{reg}^B(\beta)\big|_{\mathcal{U}}$ • $\mathrm{reg}^B(\beta) \sim L^*(H^{n-1}(F_n^d), 0)$

Assumption $(*)$. $\exists\, \beta \in CH^n(F_n^d, n) = H^n_{\mathcal{M}}(F_n^d, \mathbb{Q}(n))$ s.t.

$$\mathrm{reg}^M(\alpha_0)\big|_{\mathcal{U}} = \mathrm{reg}^B(\beta)\big|_{\mathcal{U}}.$$

Consider the simplest case $d=2$, i.e.

$$x_1^2 + x_2^2 + \cdots + x_n^2 = x_0^2 \qquad \text{in } \mathbb{P}^n.$$

Thm (A'Campo – H.). Under assumption $(*)$, $n \geq 3$ odd:

$$\tilde{F}(\underbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}_{n \text{ times}}) \sim_{\mathbb{Q}^\times} \begin{cases} \pi^{4k} \zeta^*(-2k) \sim \pi^{2k} \zeta(2k+1) & n = 4k+1, \\ \pi^{4k-2} L^*(\chi, -2k+1) \sim \pi^{2k-1} L(\chi, 2k) & n = 4k-1. \\ \quad \chi = \left(\frac{\bullet}{-4}\right) & \end{cases}$$

Examples Numerically (up to 15 digits):

(1) $n=3$: $\tilde{F}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = 8\pi^2 L'(\chi, -1) = 4\pi^1 L(\chi, 2)$

(2) $n=5$: $\tilde{F}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = -\tfrac{14}{3}\pi^4 \zeta'(-2) = \tfrac{7}{6}\pi^2 \zeta(3)$

(Apery's constant)

Sketch of proof.

Step 1. In the cat. of Chow motives $/\mathbb{Q}$:

$$M \cong \begin{cases} \mathbb{Q}(-2k) & n = 4k+1, \\ \mathbb{Q}(\chi)(-2k+1) & n = 4k-1. \end{cases}$$

Step 2. ($n = 4k+1$ for simplicity)

$$\begin{aligned} H^n_{\mathcal{M}}(M, n) &= H^n_{\mathcal{M}}(\mathbb{Q}(-2k), n) \\ &= H^{n-4k}_{\mathcal{M}}(\mathbb{Q}, n-2k) \\ &= H^1_{\mathcal{M}}(\mathbb{Q}, 2k+1) \\ &= K_{4k+1}(\mathbb{Q}) \end{aligned}$$

& Borel proved:
- $\operatorname{rk} K_{4k+1}(\mathbb{Q}) = 1$
- $\zeta^*(-2k) \sim$ regulator of $K_{4k+1}(\mathbb{Q})$.

$\square$

3. What's next?

- Can we remove assumption (*)?

- For $d > 2$, should get consequences for Beilinson's conjecture for characters of $\mathbb{Q}(\zeta_d)$.
- In higher rank: get more classes by going to covers & pushing forward.

- Finally, deformation to $t \neq 0$ & applications to automorphic forms.